

Supremum of the function $S_1(t)$ on short intervals

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Abstract

We prove a lower bound on the supremum of the function $S_1(T)$ on short intervals, defined by the integration of the argument of the Riemann zeta-function. The same type of result on the supremum of $S(T)$ have already been obtained by Karatsuba and Korolev. Our result is based on the idea of the paper of Karatsuba and Korolev. Also, we show an improved Omega-result for a lower bound.

1 Introduction

We consider the argument of the Riemann zeta function $\zeta(s)$, where $s = \sigma + ti$ is a complex variable, on the critical line $\sigma = \frac{1}{2}$.

We introduce the functions $S(t)$ and $S_1(t)$. When T is not the ordinate of any zero of $\zeta(s)$, we define

$$S(T) = \frac{1}{\pi} \arg \zeta \left(\frac{1}{2} + Ti \right).$$

This is obtained by continuous variation along the straight lines connecting 2 , $2 + Ti$, and $\frac{1}{2} + Ti$, starting with the value zero. When T is the ordinate of some zero of $\zeta(s)$, we define

$$S(T) = \frac{1}{2} \{S(T+0) + S(T-0)\}.$$

Next, we define $S_1(T)$ by

$$S_1(T) = \int_0^T S(t) dt + C,$$

where C is the constant defined by

$$C = \frac{1}{\pi} \int_{\frac{1}{2}}^{\infty} \log |\zeta(\sigma)| d\sigma.$$

It is a classical result of von Mangoldt (cf. chapter 9 of Titchmarsh [14]) that there exists a number $T_0 > 0$ such that for $T > T_0$ we have

$$S(T) = O(\log T).$$

Also, Littlewood [9] proved that there exists a number $T_0 > 0$ such that for $T > T_0$ we have

$$S_1(T) = O(\log T).$$

Further, Littlewood proved that under the Riemann Hypothesis we have

$$S(T) = O\left(\frac{\log T}{\log \log T}\right)$$

and

$$S_1(T) = O\left(\frac{\log T}{(\log \log T)^2}\right).$$

There exist some known results for $S(t)$ on short intervals. In 1946, Selberg [12] proved the inequalities

$$\sup_{T \leq t \leq 2T} (\pm S(t)) \geq A \frac{(\log T)^{\frac{1}{3}}}{(\log \log T)^{\frac{7}{3}}},$$

where A is a positive absolute constant. Also, a similar result for $S_1(t)$ is

$$S_1(t) = \Omega_{\pm} \left(\frac{(\log t)^{\frac{1}{3}}}{(\log \log t)^{\frac{10}{3}}} \right). \quad (1)$$

Also, Tsang [15] proved for $S_1(t)$ that

$$S_1(t) = \begin{cases} \Omega_+ \left(\frac{(\log t)^{\frac{1}{2}}}{(\log \log t)^{\frac{9}{4}}} \right) & \text{unconditionally,} \\ \Omega_- \left(\frac{(\log t)^{\frac{1}{3}}}{(\log \log t)^{\frac{4}{3}}} \right) & \text{unconditionally,} \\ \Omega_{\pm} \left(\frac{(\log t)^{\frac{1}{2}}}{(\log \log t)^{\frac{5}{2}}} \right) & \text{assuming } R.H. \end{cases} \quad (2)$$

In 1977, Montgomery [10] established the following result under the assumption of the Riemann hypothesis: In the interval $(T^{\frac{1}{6}}, T)$, there exist points t_0 and t_1 such that

$$(-1)^j S(t_j) \geq \frac{1}{20} \left(\frac{\log T}{\log \log T} \right)^{\frac{1}{2}}, \quad j = 0, 1.$$

In 1986, Tsang [15] improved the methods of [12] to obtain the following inequalities strengthening the above results of Selberg and Montgomery:

$$\sup_{T \leq t \leq 2T} (\pm S(t)) \geq A \left(\frac{\log T}{\log \log T} \right)^a,$$

where $A > 0$ is an absolute constant and the value of a is equal to $\frac{1}{2}$ if the Riemann hypothesis is true and equal to $\frac{1}{3}$ otherwise.

In 2005, Karatsuba and Korolev [6] established the following result: Let $0 < \epsilon < \frac{1}{10^3}$, $T \geq T_0(\epsilon) > 0$, and $H = T^{\frac{27}{82} + \epsilon}$. Then

$$\sup_{T-H \leq t \leq T+2H} (\pm S(t)) \geq \frac{\epsilon^{\frac{5}{4}}}{1000} \left(\frac{\log T}{\log \log T} \right)^{\frac{1}{3}}.$$

Our result in the present paper is obtained by applying the method of proving the above result to the function $S_1(t)$.

Theorem 1. @

Let $0 < \epsilon < \frac{1}{10^3}$, $T \geq T_0(\epsilon) > 0$, and $H = T^{\frac{27}{82} + \epsilon}$. Then

$$\sup_{T-H \leq t \leq T+2H} (\pm S_1(t)) \geq \frac{\epsilon}{4000\pi} \left(\frac{(\log T)^{\frac{1}{3}}}{(\log \log T)^{\frac{5}{3}}} \right).$$

This can be proven similarly to the above result of Karatsuba and Korolev [6]. So in this paper, we describe just the outline of the proof of Theorem 1. However, lemmas to apply for the proof of Theorem 1 are different from those in [6]. There are five lemmas to apply, four lemmas among them are different. Therefore, we describe the details of the proofs of those lemmas, which are Lemma 1, Lemma 2, Lemma 3 and Lemma 4. The basic ideas of the proofs of Lemmas 1, 2, 3 and 4 are based on the proof of Theorem 2, Lemma 2, Lemma 4 and Lemma 3, respectively, of Chapter 3 in Karatsuba and Korolev [6].

Theorem 2.

$$S_1(t) = \Omega_{\pm} \left(\frac{(\log t)^{\frac{1}{3}}}{(\log \log t)^{\frac{5}{3}}} \right).$$

Theorem 2 can be seen immediately from Theorem 1. This is an improvement of Selberg's result (1). Moreover, for Ω_+ , Theorem 2 is also an improvement of Tsang's result (2).

There are functions $S_2(t), S_3(t), \dots$ defined by

$$S_m(t) = \int_0^t S_{m-1}(u) du + C_m$$

for $m \geq 2$, where constants C_m depend on m . It seems that we cannot apply the method in Karatsuba and Korolev [6] for $S_2(t), S_3(t)$, etc. because $S_2(t)$, etc. do not have the expression like

$$S_1(t) = \frac{1}{\pi} \int_{\frac{1}{2}}^{\frac{3}{2}} \log |\zeta(\sigma + ti)| d\sigma + O(1) \quad (3)$$

for $S_1(t)$ in p. 274 of Selberg [13]. This expression is essential in the proof of Lemma 1. The basic idea of the method in Karatsuba and Korolev [6] relies on Lemma 1. Therefore, the method in this paper cannot be applied to $S_2(t)$, etc.

Therefore, some new idea or the expression like (3) will be necessary to obtain the result similar to our Theorem 1, for functions $S_2(t)$, etc.

2 Some lemmas

Here we introduce the following notations.

Let $2 \leq x \leq t^2$. We set

$$\sigma_{x,t} = \frac{1}{2} + 2 \max \left(\left| \beta - \frac{1}{2} \right|, \frac{1}{\log x} \right),$$

where β ranges over the real parts of the zeros $\rho = \beta + \gamma i$ of the Riemann zeta function that satisfy the condition

$$|\gamma - t| \leq \frac{x^{3|\beta - \frac{1}{2}|}}{\log x}.$$

Also, we set

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ with a prime } p \text{ and an integer } k \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Using these notations, we state the following lemmas.

Lemma 1. @

Let $f(z)$ be a function taking real values on the real line, analytic on the strip $|\Im z| \leq 1$, and satisfying the inequality $|f(z)| \leq c(|z| + 1)^{-(1+\alpha)}$, $c > 0$, $\alpha > 0$, on this strip. Then, the formula

$$\begin{aligned} \int_{-\infty}^{\infty} f(u) S_1(t+u) du &= \frac{1}{\pi} \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^{\frac{1}{2}} (\log n)^2} \Re \left(\frac{1}{n^{ti}} \hat{f}(\log n) \right) - C \hat{f}(0) \\ &\quad + 2 \left\{ \sum_{\beta > \frac{1}{2}} \int_{\frac{1}{2}}^{\beta} \int_0^{\beta - \sigma} \Re f(\gamma - t - xi) dx d\sigma \right. \\ &\quad \left. - \int_{\frac{1}{2}}^1 \int_0^{1-\sigma} \Re f(-t - xi) dx d\sigma \right\}, \end{aligned}$$

where $\hat{f}(x)$ is given by the formula

$$\hat{f}(x) = \int_{-\infty}^{\infty} f(u) e^{-ixu} du,$$

holds for any t , where the summation in the last sum is taken over all complex zeros $\rho = \beta + \gamma i$ of $\zeta(s)$ to the right of the critical line, and where

$$C = \frac{1}{\pi} \int_{\frac{1}{2}}^{\infty} \log |\zeta(\sigma)| d\sigma.$$

Lemma 2. @

For any sufficiently large positive values of H , t , and τ with $\tau < \log t$ and $H < t$,

$$\int_{-\frac{1}{2}H\tau}^{\frac{1}{2}H\tau} \left(\frac{\sin u}{u} \right)^2 S_1 \left(t + \frac{2u}{\tau} \right) du = W(t) + R(t) + O \left(\frac{\log t}{\tau H} \right) + O(1),$$

where

$$W(t) = \sum_{p \leq e^\tau} \frac{\cos(t \log p)}{p^{\frac{1}{2}} \log p} \left(1 - \frac{\log p}{\tau}\right),$$

$$R(t) = \tau \sum_{\beta > \frac{1}{2}} \int_{\frac{1}{2}}^{\beta} \int_0^{\beta-\sigma} \Re \left(\frac{\sin \frac{\tau}{2}(\gamma - t - xi)}{\frac{\tau}{2}(\gamma - t - xi)} \right)^2 dx d\sigma.$$

Lemma 3. @

Let ϵ with $0 < \epsilon < \frac{1}{1000}$ be fixed. Let $T \geq T_0(\epsilon) > 0$, $H = T^{\frac{27}{82} + \epsilon}$ and k be an integer such that $k \geq k_0(\epsilon) > 1$, let $m = 2k + 1$, $\tau = 2 \log \log H$, and $m\tau < \frac{1}{10}\epsilon \log T$. Then the function $R(t)$ defined by Lemma 2 satisfies the inequality

$$\int_T^{T+H} |R(t)|^m dt < H \left\{ 25^m + (\log T)^3 \left(\frac{50\tau m^2}{\epsilon^3 \log T} \right)^m \right\}.$$

Lemma 4. @

Let $T \geq T_0 > 0$, $e^2 < H < T$, $2 < \tau < \log H$, and k be an integer such that $k \geq k_0 > 1$ and $(2k \log k)^2 < e^{\frac{4}{5}\tau}$. Then

$$\int_T^{T+H} W(t)^{2k} dt > \left(\frac{1}{5\sqrt{10}e} \cdot \frac{k^{\frac{1}{2}}}{\log k} \right)^{2k} H - e^{3k\tau}, \quad (4)$$

$$\left| \int_T^{T+H} W(t)^{2k+1} dt \right| < e^{3k\tau + \frac{3}{2}\tau}. \quad (5)$$

This lemma is Lemma 3 of Chapter 3 in Karatsuba and Korolev [6]. But in Karatsuba and Korolev [6], the function $W(t)$ is defined by

$$W(t) = - \sum_{p \leq e^\tau} \frac{\sin(t \log p)}{p^{\frac{1}{2}}} \left(1 - \frac{\log p}{\tau}\right),$$

which are different from the definition in this paper.

The following lemma is given in Karatsuba and Korolev [6].

Lemma 5. @

Let $H > 0$ and $M > 0$, let $k \geq 1$ be an integer, and let $W(t)$, $R(t)$ be real functions which satisfy the conditions

- 1) $\int_T^{T+H} |W(t)|^{2k} dt \geq HM^{2k},$
- 2) $\left| \int_T^{T+H} W(t)^{2k+1} dt \right| \leq \frac{1}{2} HM^{2k+1},$
- 3) $\int_T^{T+H} |R(t)|^{2k+1} dt < H \left(\frac{M}{2} \right)^{2k+1}.$

Then

$$\max_{T \leq t \leq T+H} \pm(W(t) + R(t)) \geq \frac{1}{8}M.$$

This lemma is Lemma 1 of Chapter 3 in Karatsuba and Korolev [6].

3 Proof of Lemma 1

This proof is an analogue of the proof of Theorem 2 of Chapter 3 in Karatsuba and Korolev [6].

Proof. Put $\frac{1}{2} \leq \sigma \leq \frac{3}{2}$. We set $\psi(z) = f((\sigma - z)i - t)$ and take $X \geq 2(|t| + 10)$ such that the distance from the ordinate of any zero of $\zeta(s)$ to X is not less than $c(\log X)^{-1}$, where c is a positive absolute constant.

Let Γ be the boundary of the rectangle with the vertices $\sigma \pm Xi$, $\frac{3}{2} \pm Xi$, and let a horizontal cut be drawn from the line $\Re s = \sigma$ inside this rectangle to each zero $\rho = \beta + \gamma i$ and also to the point $z = 1$. Then the functions $\log \zeta(z)$ and $\psi(z)$ are analytic inside Γ .

By the residue theorem, the following equality holds:

$$\begin{aligned} 0 &= \int_{\Gamma} \psi(z) \log \zeta(z) dz \\ &= \left(\int_{\frac{3}{2}-Xi}^{\frac{3}{2}+Xi} - \int_{\sigma+Xi}^{\frac{3}{2}+Xi} - \int_{\sigma-Xi}^{\sigma+Xi} + \int_{\sigma-Xi}^{\frac{3}{2}-Xi} \right) \psi(z) \log \zeta(z) dz \\ &= I_1 - I_2 - I_3 + I_4, \end{aligned}$$

say. Then, we have

$$I_1 = i \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^{\sigma+ti} \log n} \hat{f}(\log n) + O\left(\frac{1}{X^{\alpha}}\right)$$

since for $\alpha = \frac{3}{2} - \sigma$

$$\begin{aligned} \int_{-\infty}^{\infty} \psi\left(\frac{3}{2} + ui\right) \log \zeta\left(\frac{3}{2} + ui\right) du &= \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^{\frac{3}{2}} \log n} \int_{-\infty}^{\infty} \frac{1}{n^{ui}} f(u - t - \alpha i) du \\ &= \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^{\sigma+ti} \log n} \hat{f}(\log n). \end{aligned}$$

Also,

$$\begin{aligned} I_2 &= O\left(\frac{(\log X)^2}{X^{(1+\alpha)}}\right), \\ I_4 &= O\left(\frac{(\log X)^2}{X^{(1+\alpha)}}\right) \end{aligned}$$

as in p. 461 of Karatsuba and Korolev [6].

We denote by L the cut going from the point $\sigma + \gamma i$ to the each points $\beta + \gamma i$, and denote by $I(L)$ the integral over the banks of this cut. Then,

$$I(L) = \int_L \psi(z) \log \zeta(z) dz = 2\pi i \sum_{\beta > \sigma} \int_0^{\beta - \sigma} f(\gamma - t - xi) dx$$

as in p. 462 of Karatsuba and Korolev [6].

If L is the cut going to the point $z = 1$, then

$$I(L) = -2\pi i \int_0^{1-\sigma} f(-t - xi) dx.$$

Hence, we have

$$\begin{aligned} I_3 &= \int_{\sigma - Xi}^{\sigma + Xi} \psi(z) \log \zeta(z) dz \\ &\quad - 2\pi i \left(\sum_{\beta > \sigma} \int_0^{\beta - \sigma} f(\gamma - t - xi) dx - \int_0^{1-\sigma} f(-t - xi) dx \right). \end{aligned}$$

When X tends to infinity, we obtain

$$\begin{aligned} \lim_{X \rightarrow \infty} \int_{\sigma - Xi}^{\sigma + Xi} \psi(z) \log \zeta(z) dz &= i \int_{-\infty}^{\infty} f(u) \log \zeta(\sigma + (t + u)i) du \\ &= i \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^{\sigma + ti} \log n} \cdot \hat{f}(\log n) \\ &\quad + 2\pi i \left(\sum_{\beta > \sigma} \int_0^{\beta - \sigma} f(\gamma - t - xi) dx - \int_0^{1-\sigma} f(-t - xi) dx \right). \end{aligned}$$

Dividing by i , we get for $\sigma \geq \frac{1}{2}$

$$\begin{aligned} \int_{-\infty}^{\infty} f(u) \log \zeta(\sigma + (t + u)i) du &= \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^{\sigma + ti} \log n} \hat{f}(\log n) \\ &\quad + 2\pi \left(\sum_{\beta > \sigma} \int_0^{\beta - \sigma} f(\gamma - t - xi) dx - K(\sigma) \int_0^{1-\sigma} f(-t - xi) dx \right), \end{aligned}$$

where

$$K(\sigma) = \begin{cases} 1 & \text{for } \frac{1}{2} \leq \sigma \leq 1, \\ 0 & \text{for } \sigma > 1. \end{cases}$$

Here, taking the real part and applying (3) and integrating in σ over the interval $[\frac{1}{2}, \frac{3}{2}]$, we have

$$\begin{aligned}
& \int_{-\infty}^{\infty} \int_{\frac{1}{2}}^{\frac{3}{2}} f(u) \log |\zeta(\sigma + (t+u)i)| d\sigma du \\
&= \pi \int_{-\infty}^{\infty} S_1(t+u) f(u) du + \pi \int_{-\infty}^{\infty} f(u) C du \\
&= \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^{\frac{1}{2}} (\log n)^2} \Re \left(\frac{1}{n^{ti}} \hat{f}(\log n) \right) \\
&\quad + 2\pi \left(\int_{\frac{1}{2}}^{\frac{3}{2}} \sum_{\beta > \sigma} \int_0^{\beta-\sigma} \Re f(\gamma - t - xi) dx d\sigma - \int_{\frac{1}{2}}^{\frac{3}{2}} \int_0^{1-\sigma} \Re f(-t - xi) dx d\sigma \right).
\end{aligned}$$

Therefore

$$\begin{aligned}
& \int_{-\infty}^{\infty} S_1(t+u) f(u) du \\
&= \frac{1}{\pi} \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^{\frac{1}{2}} (\log n)^2} \Re \left(\frac{1}{n^{ti}} \hat{f}(\log n) \right) - C \hat{f}(0) \\
&\quad + 2 \left(\int_{\frac{1}{2}}^{\frac{3}{2}} \sum_{\beta > \sigma} \int_0^{\beta-\sigma} \Re f(\gamma - t - xi) dx d\sigma - \int_{\frac{1}{2}}^1 \int_0^{1-\sigma} \Re f(-t - xi) dx d\sigma \right) \\
&= \frac{1}{\pi} \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^{\frac{1}{2}} (\log n)^2} \Re \left(\frac{1}{n^{ti}} \hat{f}(\log n) \right) - C \hat{f}(0) \\
&\quad + 2 \left(\sum_{\beta > \frac{1}{2}} \int_{\frac{1}{2}}^{\beta} \int_0^{\beta-\sigma} \Re f(\gamma - t - xi) dx d\sigma - \int_{\frac{1}{2}}^1 \int_0^{1-\sigma} \Re f(-t - xi) dx d\sigma \right).
\end{aligned}$$

□

4 Proof of Lemma 2

This proof is an analogue of the proof of Lemma 2 of Chapter 3 in Karatsuba and Korolev [6].

Proof. Put $f(z) = \left(\frac{\sin \frac{\pi z}{\tau}}{\frac{\pi z}{\tau}} \right)^2$. By

$$\hat{f}(x) = \int_{-\infty}^{\infty} e^{-xui} f(u) du = \frac{2\pi}{\tau} \max \left(0, 1 - \left| \frac{x}{\tau} \right| \right),$$

we get

$$\hat{f}(\log n) = \begin{cases} \frac{2\pi}{\tau} \left(1 - \frac{\log n}{\tau} \right) & (1 \leq n \leq e^\tau) \\ 0 & (n > e^\tau) \end{cases}.$$

Then, we have

$$\begin{aligned}
\int_{-\infty}^{\infty} \left(\frac{\sin \frac{\tau u}{2}}{\frac{\tau u}{2}} \right)^2 S_1(t+u) du &= \frac{1}{\pi} \sum_{n \leq e^\tau} \frac{\Lambda(n)}{n^{\frac{1}{2}} (\log n)^2} \cdot \frac{2\pi}{\tau} \left(1 - \frac{\log n}{\tau} \right) \cos(t \log n) \\
&\quad + 2 \left\{ \sum_{\beta > \frac{1}{2}} \int_{\frac{1}{2}}^{\beta} \int_0^{\beta-\sigma} \Re \left(\frac{\sin \frac{\tau}{2}(\gamma - t - \xi i)}{\frac{\tau}{2}(\gamma - t - \xi i)} \right)^2 d\xi d\sigma \right. \\
&\quad \left. - \int_{\frac{1}{2}}^1 \int_0^{1-\sigma} \Re \left(\frac{\sin \frac{\tau}{2}(\gamma - t - \xi i)}{\frac{\tau}{2}(\gamma - t - \xi i)} \right)^2 d\xi d\sigma \right\} \\
&\quad - C \hat{f}(0)
\end{aligned} \tag{6}$$

by Lemma 1. Since for $0 \leq \xi \leq 1 - \sigma$

$$2 \left| \frac{\sin \frac{\tau}{2}(t + \xi i)}{\frac{\tau}{2}(t + \xi i)} \right|^2 < \frac{1}{5\tau}$$

as in p. 473 of Karatsuba and Korolev [6], we have

$$\left| 2 \int_{\frac{1}{2}}^1 \int_0^{1-\sigma} \Re \left(\frac{\sin \frac{\tau}{2}(\gamma - t - \xi i)}{\frac{\tau}{2}(\gamma - t - \xi i)} \right)^2 d\xi d\sigma \right| = O\left(\frac{1}{\tau}\right).$$

In the first term of the right-hand side in (6), we single out the terms corresponding to the $n = p$ in the sum and estimate the remainder terms. Then, we have

$$\sum_{2 \leq k} \sum_{p^k \leq e^\tau} \frac{\Lambda(p^{\frac{k}{2}}) \cos(t \log p^k)}{p^{\frac{k}{2}} (\log p^k)^2} \cdot \frac{2}{\tau} \left(1 - \frac{\log p^k}{\tau} \right) < \sum_{2 \leq k} \sum_{p^k \leq e^\tau} \frac{\log p}{p^{\frac{k}{2}} (\log p^k)^2} \cdot \frac{2}{\tau} \ll \frac{1}{\tau}.$$

Hence

$$\begin{aligned}
\int_{-\infty}^{\infty} \left(\frac{\sin \frac{\tau u}{2}}{\frac{\tau u}{2}} \right)^2 S_1(t+u) du &= \frac{2}{\tau} \sum_{p \leq e^\tau} \frac{\cos(t \log p)}{p^{\frac{1}{2}} \log p} \left(1 - \frac{\log p}{\tau} \right) - C \cdot \frac{2\pi}{\tau} \\
&\quad + 2 \sum_{\beta > \frac{1}{2}} \int_{\frac{1}{2}}^{\beta} \int_0^{\beta-\sigma} \Re \left(\frac{\sin \frac{\tau}{2}(\gamma - t - \xi i)}{\frac{\tau}{2}(\gamma - t - \xi i)} \right)^2 d\xi d\sigma \\
&\quad + O\left(\frac{1}{\tau}\right).
\end{aligned} \tag{7}$$

Put $v = \frac{\tau u}{2}$. Then the left-hand side of the above is equal to

$$\left(\int_{-\frac{1}{2}H\tau}^{\frac{1}{2}H\tau} + \int_{-\infty}^{-\frac{1}{2}H\tau} + \int_{\frac{1}{2}H\tau}^{\infty} \right) \left(\frac{\sin v}{v} \right)^2 S_1 \left(t + \frac{2v}{\tau} \right) \frac{2}{\tau} dv.$$

Since $S_1(t) = O(\log t)$, we have

$$\begin{aligned}
\left| \left(\int_{-\infty}^{-\frac{1}{2}H\tau} + \int_{\frac{1}{2}H\tau}^{\infty} \right) \left(\frac{\sin v}{v} \right)^2 S_1 \left(t + \frac{2v}{\tau} \right) dv \right| &\ll \frac{1}{\tau} \int_H^{\infty} \log(t + v') \frac{1}{v'^2} dv' \\
&\ll \frac{1}{\tau} \left\{ \int_H^t \frac{\log t}{v'^2} dv' + \int_t^{\infty} \frac{\log v'}{v'^2} dv' \right\} \\
&\ll \frac{1}{\tau} \left(\frac{\log t}{H} + \frac{\log t}{t} \right) \ll \frac{\log t}{\tau H}.
\end{aligned}$$

Inserting these estimates into (7) and dividing by $\frac{2}{\tau}$ the both sides, we obtain the result. \square

5 Proof of Lemma 3

This proof is an analogue of the proof of Lemma 4 of Chapter 3 in Karatsuba and Korolev [6].

Proof. We put

$$L_k = \int_T^{T+H} |R(t)|^{2k+1} dt$$

and note the inequality

$$\left| \Re \left(\frac{\sin(x - yi)}{x - yi} \right)^2 \right| < \frac{8ye^{2y}}{1 + x^2 + y^2}$$

for any $x, y \in \mathbb{R}$, $y \geq 0$ similarly to pp. 476 – 477 of Karatsuba and Korolev [6]. Then,

$$\begin{aligned}
|R(t)| &\leq \tau \left| \sum_{\substack{\gamma \\ \beta > \frac{1}{2}}} \int_{\frac{1}{2}}^{\beta} \int_0^{\beta-\sigma} \Re \left(\frac{\sin \frac{\tau}{2}(\gamma - t - \xi i)}{\frac{\tau}{2}(\gamma - t - \xi i)} \right)^2 d\xi d\sigma \right| \\
&\leq \tau \sum_{\substack{\gamma \\ \beta > \frac{1}{2}}} \int_{\frac{1}{2}}^{\beta} \int_0^{\beta-\sigma} \frac{8 \cdot \frac{\tau\xi}{2} e^{\tau\xi}}{1 + \left\{ \frac{\tau}{2}(\gamma - t) \right\}^2 + \left(\frac{\tau\xi}{2} \right)^2} d\xi d\sigma \\
&< 4\tau^2 \sum_{\substack{\gamma \\ \beta > \frac{1}{2}}} \int_{\frac{1}{2}}^{\beta} \int_0^{\beta-\frac{1}{2}} \frac{\xi e^{\tau(\beta-\frac{1}{2})}}{1 + \left\{ \frac{\tau}{2}(\gamma - t) \right\}^2 + \left(\frac{\tau}{2}(\beta - \frac{1}{2}) \right)^2} d\xi d\sigma \\
&= 8 \sum_{\substack{\gamma \\ \beta > \frac{1}{2}}} \left(\beta - \frac{1}{2} \right)^3 \frac{e^{\tau(\beta-\frac{1}{2})}}{\left(\frac{2}{\tau} \right)^2 + (\gamma - t)^2 + \left(\beta - \frac{1}{2} \right)^2}.
\end{aligned}$$

We split the last sum into two sums. The first sum Σ_1 is the sum of the terms satisfying $|\gamma - t| > (\log T)^2$, and the second sum Σ_2 is the sum of the other terms.

Here, we denote by θ_t the largest difference of the form $\beta - \frac{1}{2}$ for zeros $\rho = \beta + \gamma i$ in the rectangle $\frac{1}{2} < \beta \leq 1$, $|\gamma - t| \leq (\log T)^2$. Also, we denote by θ'_t the supremum of the form $\beta - \frac{1}{2}$ for zeros $\rho = \beta + \gamma i$ in the rectangle $\frac{1}{2} < \beta \leq 1$, $|\gamma - t| > (\log T)^2$.

As in p. 478 of Karatsuba and Korolev [6], we apply the estimation related to $\sigma_{x,t}$ and the result $N(t+1) - N(t) < 18 \log t$ which is obtained by the Riemann-von Mangoldt formula and $|S(t)| < 8 \log t$ for $t \geq t_0 > 0$. Then we take $x = (\log T)^{\frac{1}{2}}$, and we have

$$\begin{aligned} \Sigma_1 &< \left(\beta - \frac{1}{2}\right) \sum_{|\gamma-t| > (\log T)^2} \frac{2e^{\frac{\tau}{2}}}{(\gamma-t)^2} < \frac{2}{3} \theta'_t \log T \sum_{|\gamma-t| > (\log T)^2} \frac{1}{n^2} \sum_{n < |\gamma-t| \leq n+1} 1 \\ &< \frac{2}{3} \theta'_t \log T \cdot 36 \sum_{|\gamma-t| > (\log T)^2} \frac{\log T + \log n}{n^2} < 25 \theta'_t \end{aligned}$$

and

$$\begin{aligned} \Sigma_2 &< 8\theta^3 e^{\tau\theta} \sum_{|\gamma-t| \leq (\log T)^2} \frac{1}{\left(\frac{2}{\tau}\right)^2 + (\gamma-t)^2 + \left(\beta - \frac{1}{2}\right)^2} \\ &< 8\theta^3 e^{\tau\theta} \sum_{\rho} \frac{1}{(\sigma_{x,t} - \beta)^2 + (\gamma-t)^2} < 8\theta^3 e^{\tau\theta} \frac{13}{5} \cdot \frac{1}{\sigma_{x,t} - \frac{1}{2}} \log T \\ &\leq 8\theta^3 e^{\tau\theta} \frac{13}{5} \cdot \frac{5\tau}{39} \log T = 8\theta^3 e^{\tau\theta} \cdot \frac{\tau}{3} \log T. \end{aligned}$$

From the definitions of θ_t and θ'_t , we get $\theta_t < \frac{1}{2}$ and $\theta'_t < \frac{1}{2}$. Hence, we have

$$|R(t)| < 25 \left(\theta'_t + \frac{7}{2} \theta_t^3 e^{\tau\theta_t} \tau \log T \right) < \frac{25}{2} \left(1 + \frac{7}{2} \theta_t^2 e^{\tau\theta_t} \tau \log T \right).$$

Hence

$$L_k < \left(\frac{25}{2}\right)^m \int_T^{T+H} \left(1 + \frac{7}{2} \theta_t^2 e^{\tau\theta_t} \tau \log T \right)^m dt.$$

This integrand is the same as that in p. 479 of Karatsuba and Korolev [6]. Hence the estimation of the last integral is the same as in pp. 480 – 481 of Karatsuba and Korolev [6]. Along that way, we have

$$\begin{aligned} L_k &< 25^m H \left\{ 1 + \frac{24}{5} \cdot \frac{1}{m} (\log T)^3 (2m)! \left(\frac{7}{2} \tau \log T \right)^m \left(\frac{\epsilon}{10} \log T \right)^{-2m} \right\} \\ &< 25^m H \left\{ 1 + (\log T)^3 \left(\frac{2m^2 \tau}{\epsilon^3 \log T} \right)^m \right\} \\ &< H \left(25^m + (\log T)^3 \left(\frac{50m^2 \tau}{\epsilon^3 \log T} \right)^m \right). \end{aligned}$$

□

6 Proof of Lemma 4

This proof is an analogue of the proof of Lemma 3 of Chapter 3 in Karatsuba and Korolev [6].

Proof. As in pp. 474 – 475 of Karatsuba and Korolev [6], we can write

$$\int_T^{T+H} W(t)^{2k} dt = I_k = \binom{2k}{k} \frac{H}{2^{2k}} \Sigma + \theta e^{3k\tau},$$

where

$$\Sigma = \sum_{\substack{p_1 \cdots p_k = q_1 \cdots q_k \\ p_1, \dots, p_k \leq e^\tau}} f(p_1)^2 \cdots f(p_k)^2, \quad f(p) = \frac{1}{p^{\frac{1}{2}} \log p} \left(1 - \frac{\log p}{\tau} \right).$$

Then,

$$\begin{aligned} \Sigma &\geq k! \sum_{\substack{p_1, \dots, p_k \text{ are distinct} \\ p_1, \dots, p_k \leq e^\tau}} f(p_1)^2 \cdots f(p_k)^2 \\ &\geq k! \sum_{p_1 \leq e^\tau} f(p_1)^2 \sum_{\substack{p_2 \leq e^\tau \\ p_1 \neq p_2}} f(p_2)^2 \cdots \sum_{\substack{p_k \leq e^\tau \\ p_1, \dots, p_{k-1} \neq p_k}} f(p_k)^2. \end{aligned}$$

Since $\frac{d}{dp} f(p)^2 < 0$, $f(p)^2$ is monotonically decreasing function for $p \geq 2$. Also, since $(k-1)$ th prime does not exceed $2k \log k$, the inner sum of the above inequality is greater than the same sum over $2k \log k < p_k < e^{\frac{4}{5}\tau}$. Hence the inner sum over p_k is greater than

$$\left(\frac{1}{5} \right)^2 \sum_{2k \log k < p \leq e^{\frac{4}{5}\tau}} \frac{1}{p(\log p)^2}.$$

For $(2k \log k)^2 \leq e^{\frac{4}{5}\tau}$, since

$$\begin{aligned} \sum_{U < p \leq U^2} \frac{1}{p(\log p)^2} &\geq \frac{1}{4(\log U)^2} \sum_{U < p \leq U^2} \frac{1}{p} \\ &= \frac{1}{4(\log U)^2} (\log \log U^2 - \log \log U + o(1)) > \frac{1}{8(\log U)^2}, \end{aligned}$$

the sum over p_k is greater than $\frac{1}{10} \left(\frac{1}{5} \right)^2 \frac{1}{(\log k)^2}$. Also, the same lower bound holds for the sums over p_1, p_2, \dots, p_{k-1} . Therefore, we see

$$\Sigma \geq k! \left(\frac{1}{250(\log k)^2} \right)^k \geq \sqrt{2\pi k} \left(\frac{1}{5\sqrt{10e}} \cdot \frac{k^{\frac{1}{2}}}{\log k} \right)^{2k}.$$

So,

$$I_k > H \left(\frac{1}{5\sqrt{10}e} \cdot \frac{k^{\frac{1}{2}}}{\log k} \right)^{2k} - e^{3k\tau}.$$

This is the first part of Lemma 4. The second part is proved similarly to [6]. \square

7 Outline of the proof of the Theorem 1

As described in section 1, our result can be proven similarly to Theorem 5 of Chapter 3 in Karatsuba and Korolev [6]. Therefore, we describe the outline of the proof.

Outline of the proof. Put $\tau = 2 \log \log H$. Consider the right-hand side of the inequality in the statement of Lemma 3. We see easily that

$$\frac{50\tau m^2}{\epsilon^3 \log T} < \frac{500k^2}{\epsilon^3} \cdot \frac{\log \log T}{\log T} \leq \frac{k^{\frac{1}{2}}}{\log k} \cdot \frac{500k^{\frac{3}{2}}}{\epsilon^3} \cdot \frac{(\log \log T)^2}{\log T} = \frac{k^{\frac{1}{2}}}{\log k} \delta,$$

say.

Here, putting $k = \left\lceil \frac{\epsilon^2}{1000} \left(\frac{(\log T)^{\frac{2}{3}}}{(\log \log T)^{\frac{4}{3}}} \right) \right\rceil$, we have $\delta < \frac{1}{60}$, $(2k \log k)^2 < e^{\frac{4}{5}\tau}$ and $e^{3k\tau} < H^{\frac{1}{2}}$. Hence, we can apply Lemma 3 and Lemma 4. Then we have

$$\begin{aligned} \int_T^{T+H} W(t)^{2k} dt &> HM^{2k}, \\ \left| \int_T^{T+H} W(t)^{2k+1} dt \right| &< \frac{1}{2} HM^{2k+1}, \\ \int_T^{T+H} |R(t)|^{2k+1} dt &< H \left(\frac{M}{2} \right)^{2k+1}, \end{aligned}$$

with $M = \frac{k^{\frac{1}{2}}}{30 \log k}$. Thus, we see that $W(t)$ and $R(t)$ satisfy the conditions of Lemma 5 with $M = \frac{k^{\frac{1}{2}}}{30 \log k}$. Hence there are two points t_0 and t_1 such that

$$W(t_0) + R(t_0) \geq \frac{M}{8}, \quad W(t_1) + R(t_1) \leq -\frac{M}{8}$$

in the interval $T \leq t \leq T + H$. By Lemma 2, we have

$$\begin{aligned} \int_{-\frac{1}{2}H\tau}^{\frac{1}{2}H\tau} \left(\frac{\sin u}{u} \right)^2 S_1 \left(t_0 + \frac{2u}{\tau} \right) du &\geq \frac{M}{8} + O \left(\frac{\log t_0}{\tau H} \right), \\ \int_{-\frac{1}{2}H\tau}^{\frac{1}{2}H\tau} \left(\frac{\sin u}{u} \right)^2 S_1 \left(t_1 + \frac{2u}{\tau} \right) du &\leq -\frac{M}{8} + O \left(\frac{\log t_1}{\tau H} \right). \end{aligned}$$

Here, putting

$$M_0 = \sup_{T-H \leq t \leq T+2T} S_1(t), \quad M_1 = \inf_{T-H \leq t \leq T+2T} S_1(t),$$

we have

$$\begin{aligned} \int_{-\frac{1}{2}H\tau}^{\frac{1}{2}H\tau} \left(\frac{\sin u}{u} \right)^2 S_1 \left(t_0 + \frac{2u}{\tau} \right) du &< M_0 \int_{-\infty}^{\infty} \left(\frac{\sin u}{u} \right)^2 = \frac{\pi}{2} M_0 \quad (M_0 > 0), \\ \int_{-\frac{1}{2}H\tau}^{\frac{1}{2}H\tau} \left(\frac{\sin u}{u} \right)^2 S_1 \left(t_1 + \frac{2u}{\tau} \right) du &> M_1 \int_{-\infty}^{\infty} \left(\frac{\sin u}{u} \right)^2 = \frac{\pi}{2} M_1 \quad (M_1 < 0). \end{aligned}$$

Therefore, we obtain for $r = 0, 1$

$$(-1)^r M_r > \frac{2}{\pi} \cdot \frac{M}{8} + O \left(\frac{\log t_r}{\tau H} \right) > \frac{1}{4\pi} \cdot \frac{k^{\frac{1}{2}}}{30 \log k} > \frac{\epsilon}{4000\pi} \left(\frac{(\log T)^{\frac{1}{3}}}{(\log \log T)^{\frac{5}{3}}} \right).$$

Thus, we obtain the result.

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References

- [1] A. Fujii, On the zeros of the Riemann zeta function, Comment. Math. Univ. Sancti Pauli **51**, (2002), 1-17.
- [2] A. Fujii, On the zeros of the Riemann zeta function *II*, Comment. Math. Univ. Sancti Pauli **52**, (2003), 165-190.
- [3] A. Fujii, An explicit estimate in the theory of the distribution of the zeros of the Riemann zeta function, Comment. Math. Univ. Sancti Pauli, **53**, (2004), 85-114.
- [4] A. Fujii, A note on the distribution of the argument of the Riemann zeta function, Comment. Math. Univ. Sancti Pauli, **55**, (2006), 135-147.
- [5] A. A. Karatsuba, A density thorem and the vehaviour of the argument of the Riemann zeta- function, Mat. Zametki **60**, (1990), 448-449; English transl, Math. Notes **60**, (1996), 333-334.
- [6] A. A. Karatsuba and M. A. Korolev, The argument of the Riemann zeta function, Russian Math. Serveys **60**:3, (2005), 433-488.

- [7] M. A. Korolev, On the argument of the Riemann zeta function on the critical line, *Izv. Ross. Akad. Nauk Ser. Mat.* **67**:2, (2003), 21-60; English transl., *Izv. Math.* **67** (2003), 225-264.
- [8] M. A. Korolev, On large values of the function $S(t)$ on short intervals, *Izv. Ross. Akad. Nauk Ser. Mat.* **69**:1, (2005), 115-124, English transl, *Izv. Math.* **69**, 2005, 113-122.
- [9] J. E. Littlewood, On the zeros of the Riemann zeta function, *Proc. Camb. Phil. Soc.*, 22, (1924), 295-318.
- [10] H. L. Montgomery, Extreme values of the Riemann zeta-function, *Comment. Math. Helv.*, **52**, (1977), 511-518.
- [11] A. Selberg, On the remainder in the formula for $N(T)$, the number of zeros of $\zeta(s)$ in the strip $0 < t < T$, *Avh. Norske Vir. Akad. Oslo I*:1, (1944), 1-27.
- [12] A. Selberg, Contributions to the theory of the Riemann zeta-function, *Arch. Math. Naturvid.* **48**:5, (1946), 89-155.
- [13] A. Selberg, *Collected Works*, vol I, 1989, Springer.
- [14] E. C. Titchmarsh, *The theory of the Riemann zeta-function*, Second Edition; Revised by D. R. Heath-Brown. Clarendon Press Oxford, 1986.
- [15] K. -M. Tsang, Some ω -theorems for the Riemann zeta-function, *Acta Arith.*, **46**:4, (1986), 369-395.